Assignment 4

This homework is due *Tuesday* Oct 4.

There are total 24 points in this assignment. 21 points is considered 100%. If you go over 21 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 2.4 in Bartle–Sherbert.

(1) [3pt] (2.4.6) Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A+B = \{a+b \mid a \in A, b \in B\}$. Prove that $\sup(A+B) = \sup A + \sup B$ and $\inf(A+B) = \inf A + \inf B$.

(2) [3pt] (2.4.7) Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that

$$\sup\{f(x)+g(x) \mid x \in X\} \leq \sup\{f(x) \mid x \in X\} + \sup\{g(x) \mid x \in X\}$$

and that

$$\inf\{f(x)+g(x) \mid x \in X\} \geq \inf\{f(x) \mid x \in X\} + \inf\{g(x) \mid x \in X\}.$$

Give examples to show that each of these inequalities can be either equalities or strict inequalities.

(3) (2.4.8) Let $X = Y = (0,1) \subseteq \mathbb{R}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x,y) = 2x+y$.

(a) [2pt] For each $x \in X$, find $f(x) = \sup\{h(x,y) \mid y \in Y\}$; then find $\inf\{f(x) \mid x \in X\}$.

(b) [2pt] For each $y \in Y$, find $g(y) = \inf\{h(x,y) \mid x \in X\}$; then find $\sup\{g(y) \mid y \in Y\}$. Compare with the result found in (a).

(4) [4pt] (2.4.9) Perform the computations in (a), (b) of Problem 3 for the function $h : X \times Y \to \mathbb{R}$ defined by

$$h(x,y) = \begin{cases} 0, & \text{if } x < y, \\ 1, & \text{if } x \geq y. \end{cases}$$

(5) [4pt] (2.4.10) Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be defined by

$$f(x) = \sup\{h(x,y) \mid y \in Y\}, \qquad g(y) = \inf\{h(x,y) \mid x \in X\}.$$

Prove that $\sup\{g(y) \mid y \in Y\} \leq \inf\{f(x) \mid x \in X\}$.

Comment. This inequality can be also expressed in the following way:

$$\sup_{y\in Y} \inf_{x\in X} h(x,y) \leq \inf_{x\in X} \sup_{y\in Y} h(x,y).$$

Previous two problems show that this non-strict inequality may be either an equality or a strict inequality.

— *see next page* —

(6) [3pt] (2.4.18) If $u > 0$ is any real number and $x < y$, show that there exists a rational number r such that $x < ru < y$. (Hence $u\mathbb{Q} = \{ru \mid r \in \mathbb{Q}\}$ is dense in $\mathbb{R}$.)

(7) [3pt] Prove that the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.
COMMENT. Before we proved that $\mathbb{R}$ is uncountable, we struggled to prove that even *one* irrational number exists (we proved that $\sqrt{2}$ exists in $\mathbb{R}$). After we solve this problem, we not only know that irrational numbers exist, but also that there are awful lot of them.